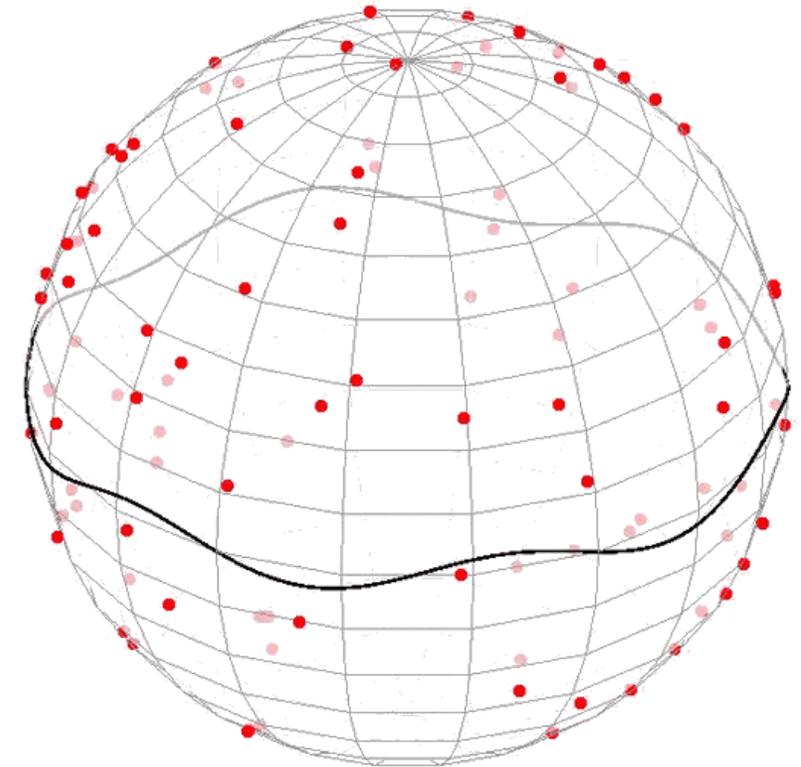


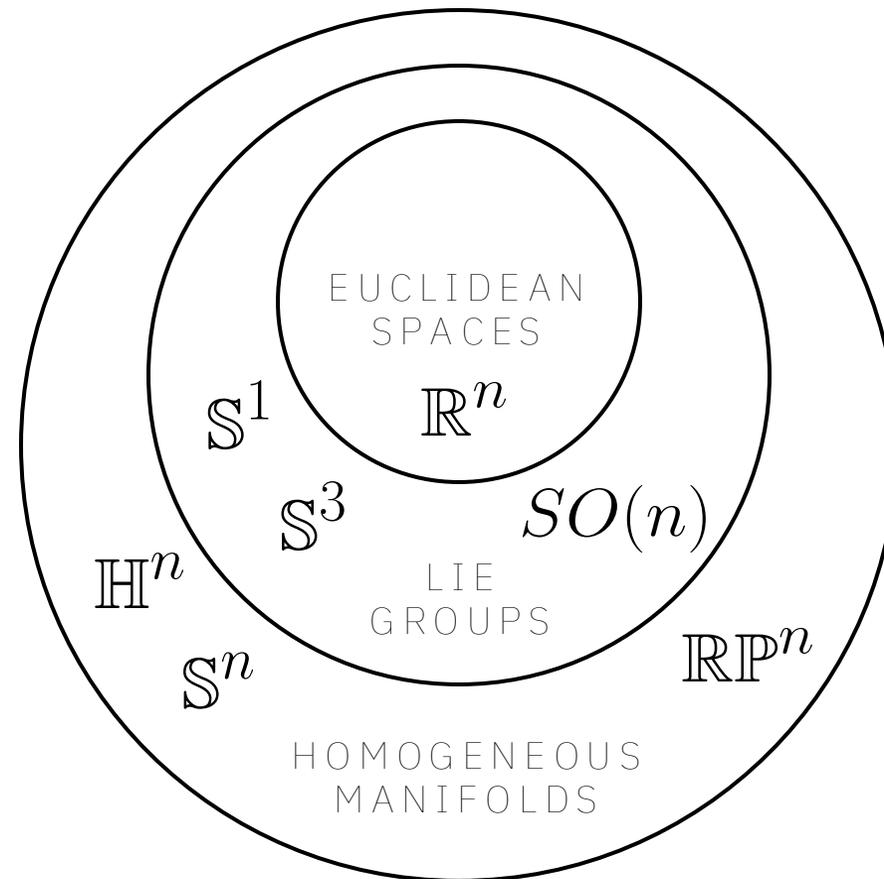
Almost Global Asymptotic Trajectory Tracking for Fully-Actuated Mechanical Systems on Homogeneous Riemannian Manifolds

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In this talk, we consider **trajectory tracking control** for **mechanical systems** evolving on **smooth manifolds**.



Fully-Actuated Mechanical Systems

from kinetic energy metric (“mass matrix”)

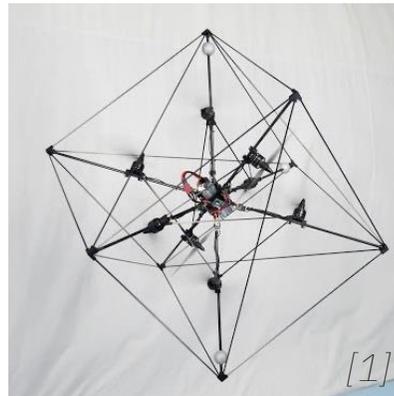
We consider fully-actuated mechanical systems of the form $\nabla_{\dot{q}}\dot{q} = f_q^\#$, where:

1. the configuration $q \in Q$ lives in a smooth, boundaryless manifold,
2. the state $\dot{q} \in TQ$ encompasses both the configuration and velocity, and
3. $f_q \in T^*Q$ is the applied control force, and may be chosen arbitrarily.

(Equivalent to the more familiar $M(q)\dot{v} + C(q, v)v = f$ often seen in robotics.)

EXAMPLES:

Entire physical systems, like the Omnicopter on $SE(3)$



[1]

Subsystems of underactuated systems, like a quadrotor’s attitude dynamics evolving on $SO(3)$



Ascending Technologies

Control Objectives for Mechanical Systems

control forces  *state*  *equilibrium* 

the **“Regulation Problem”**: Choose f_q such that $\text{dist}(\dot{q}(t), 0_{TQ}) \rightarrow 0$ as $t \rightarrow \infty$

A theorem of Koditschek solves the regulation problem in an almost global sense using a suitable choice of artificial potential energy and dissipation forces [2].

we consider the more general **“Tracking Problem”**:

Given a curve $q_d : \mathbb{R} \rightarrow Q$, choose f_q such that $\text{dist}(\dot{q}(t), \dot{q}_d(t)) \rightarrow 0$ as $t \rightarrow \infty$

reference trajectory 

Example: the Tracking Problem in Euclidean Space

Consider a forced particle $q \in \mathbb{R}^3$ and a reference trajectory: $m \ddot{q} = f, \quad q_d : \mathbb{R} \rightarrow \mathbb{R}^3$

Define “error coordinates”: $(e, \dot{e}) = (q - q_d(t), \dot{q} - \dot{q}_d(t))$

Derive “error dynamics”: $\ddot{e} = \frac{1}{m}f - \ddot{q}_d(t)$

Propose a feedback transformation: $f = m \ddot{q}_d(t) + u$

$$\implies m \ddot{e} = u := -k_P e - k_D \dot{e}$$

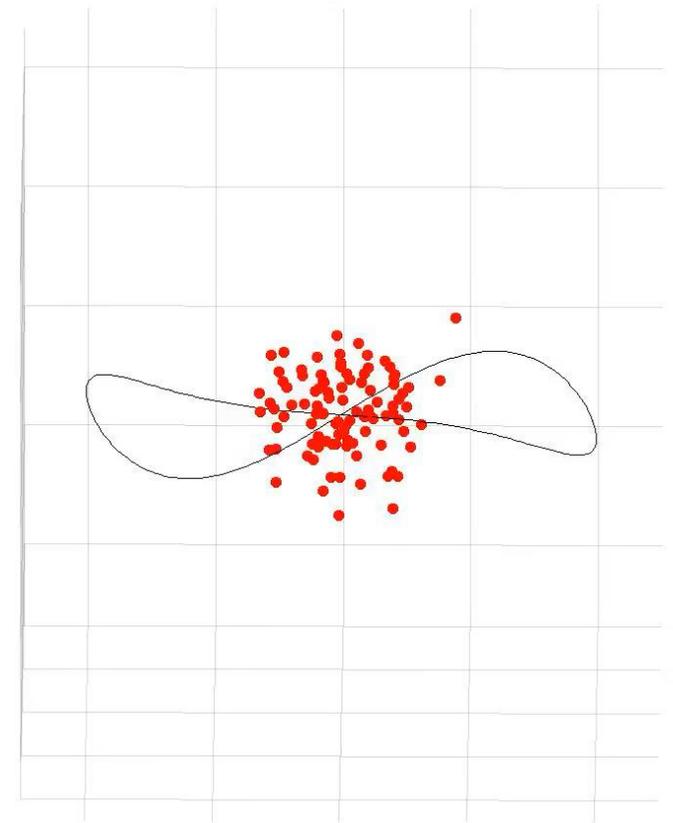
...which yields asymptotically stable error dynamics.

WHAT DID WE REALLY DO HERE?

1. We identified a *state-valued* tracking error, such that

$$(e, \dot{e}) = (0, 0) \iff (q, \dot{q}) = (q_d(t), \dot{q}_d(t))$$

2. We used feedback to transform the error dynamics into *the original system*, for which we solve *the regulation problem!*



What if our system evolves on a **manifold**, and we can't just **subtract** the actual and reference trajectories?

If $Q = G$ is a **Lie group**, no problem – just use the inverse!

Tracking Error on Lie Groups, e.g., $SO(3)$

For each element $g \in G$, there exists a *unique* element $g^{-1} \in G$ such that $g^{-1}g = 1$.

Therefore, for $e = g_d(t)^{-1}g$, it follows that $e = 1$ if and only if $g = g_d$.

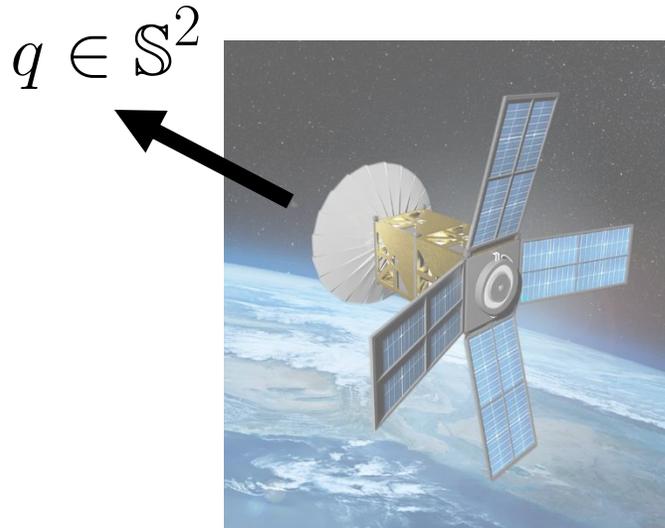
We can extend this definition to a tracking error $(e, \xi_e) \in G \times \mathfrak{g}$ such that $\dot{e} = e \xi_e$;

In fact, when $G = \mathbb{R}^n$, then $e = g_d(t)^{-1}g = g - g_d(t)$ (the tracking error from before)!

“...the tracking problem on a Lie group is more closely related to tracking on \mathbb{R}^n than it is to the general Riemannian case, for which the group operation is lacking.” [2]

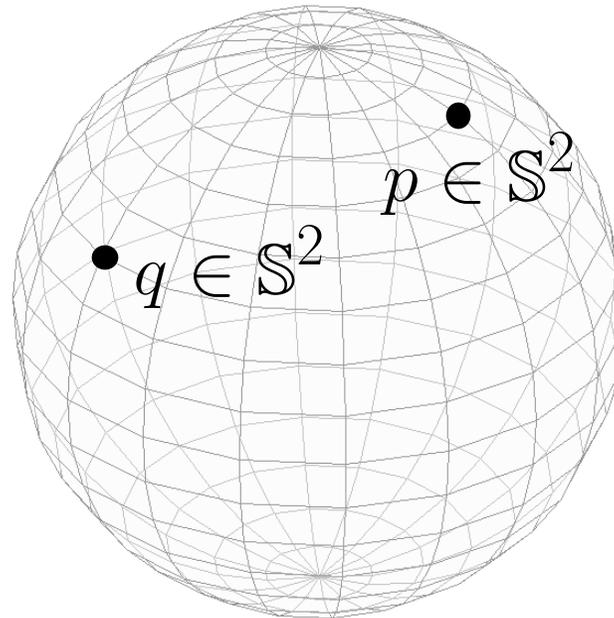
problem: not all **manifolds** are **Lie groups**!

“REDUCED-ATTITUDE”
TRACKING FOR
SATELLITES

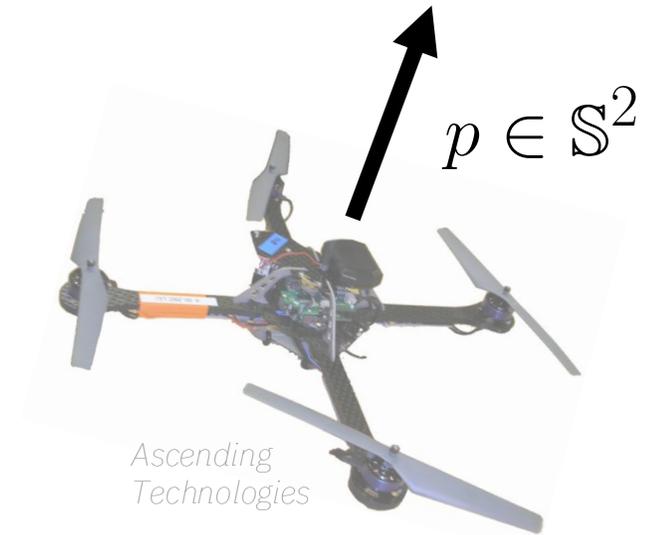


Dassault Systèmes

$$\mathbb{S}^2 = \{q \in \mathbb{R}^3 : q^T q = 1\}$$



TILT-PRIORITIZED
CONTROL FOR
QUADROTORS



fact: \mathbb{S}^n is a Lie group if and only if $n \in \{0, 1, 3\}$!

question: on which **class of manifolds** can we reduce the **tracking problem** to the **regulation problem** in an **almost global** sense?

In this work, we show how to do this on arbitrary **homogeneous Riemannian manifolds.**

Homogeneous Riemannian Manifolds

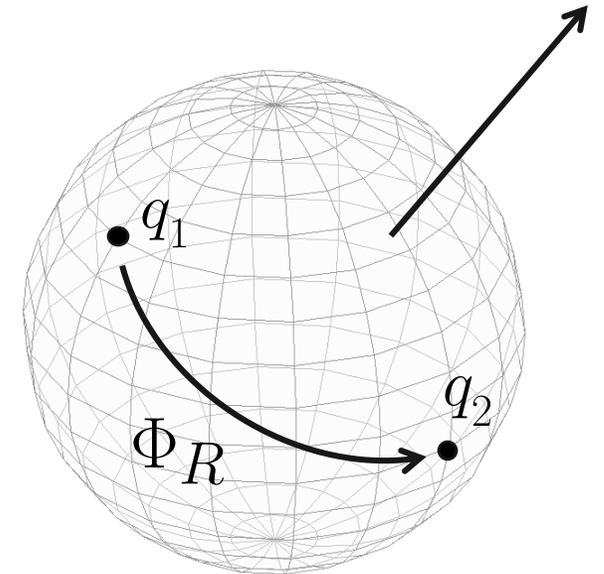
On any homogeneous Riemannian manifold (Q, Φ, κ) , we have a transitive group action $\Phi : G \times Q \rightarrow Q$ for some Lie group G . This means that for any $q_1, q_2 \in Q$, there exists some $g \in G$ such that $\Phi(g, q_1) = q_2$.

We also have a Φ -invariant Riemannian metric κ , meaning that

$$\kappa(v_q, w_q) = \kappa(d\Phi_g(v_q), d\Phi_g(w_q)) \text{ for all } g \in G \text{ and } v_q, w_q \in TQ.$$

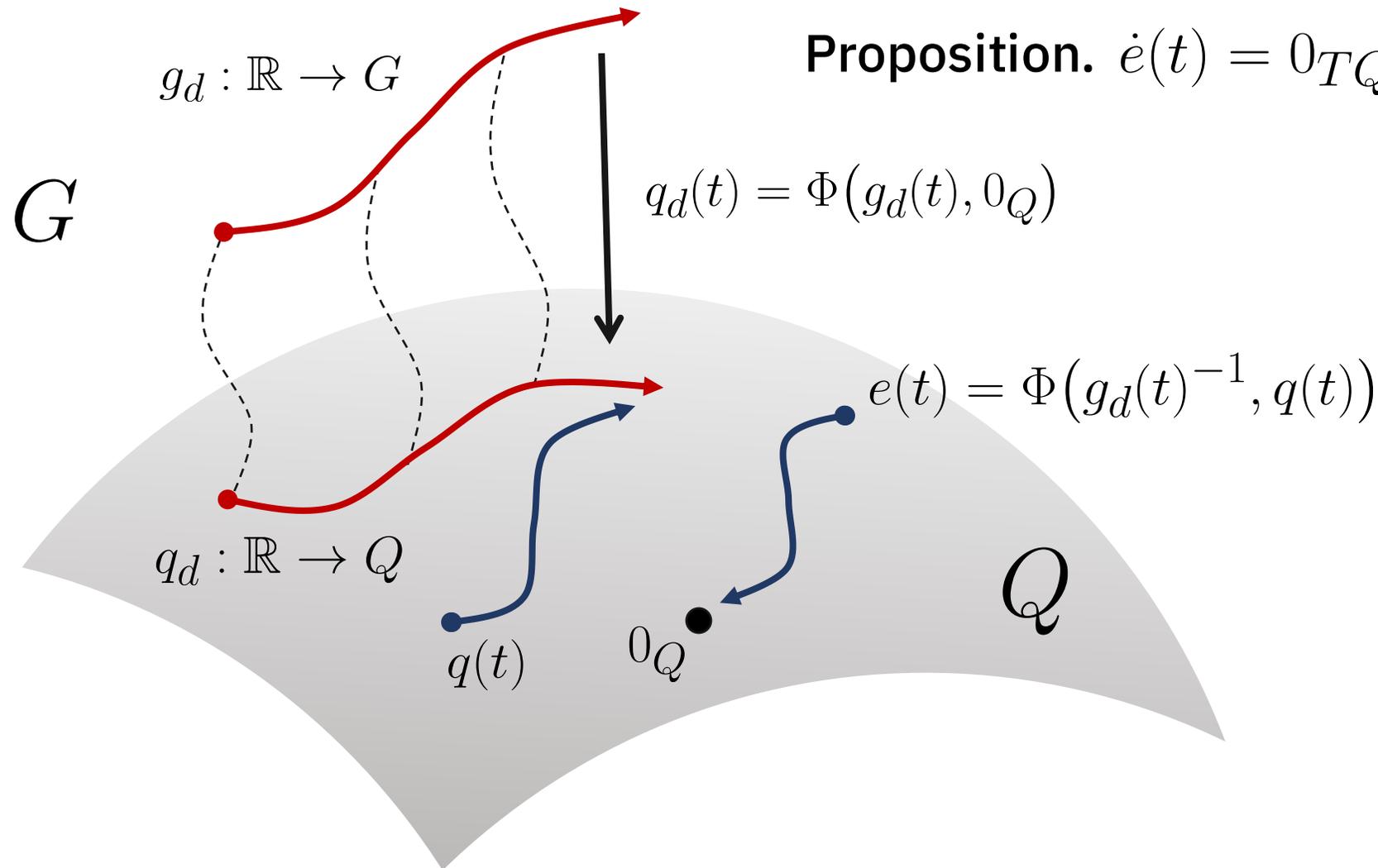
Example 1 (*Lie Groups*). For any $g_1, g_2 \in G$, $g \cdot g_1 = g_2 \implies g = g_2 \cdot g_1^{-1}$. We may define a left-invariant metric $\kappa_{\mathbb{I}}(v_g, w_g) = \mathbb{I}(dL_{g^{-1}}(v_g), dL_{g^{-1}}(w_g))$.

Example 2 (*n-Spheres*). For any $q_1, q_2 \in \mathbb{S}^n$, there exists $R \in SO(n+1)$ such that $R \cdot q_1 = q_2$. We inherit the metric from the embedding $\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$.



How can we define a **tracking error** on
a **homogeneous Riemannian manifold**?

Tracking Error on Homogeneous Riemannian Manifolds



Proposition. $\dot{e}(t) = 0_{TQ}$ if and only if $\dot{q}(t) = \dot{q}_d(t)$.

ON A LIE GROUP:

$$e = g_d(t)^{-1}g$$

IN EUCLIDEAN SPACE:

$$e = g - g_d(t)$$

Theorem (*Main Result [4]*). For a fully-actuated mechanical system on (Q, Φ, κ) , a lifted reference $g_d : \mathbb{R} \rightarrow G$ with bounded velocity, and a proper function $P : Q \rightarrow \mathbb{R}$ with a unique minimizer at 0_Q and no degenerate critical points, the control policy

$$f_q(t, \dot{q}) = -d\Phi_{g_d}^* \left(\overbrace{dP(e) + k_D \dot{e}}^{\text{FEEDBACK TERMS}} + \underbrace{(\nabla_{\dot{e}q}(\dot{e}^q + 2X\dot{q}))^b}_{\text{FEEDFORWARD TERMS}} \right)$$

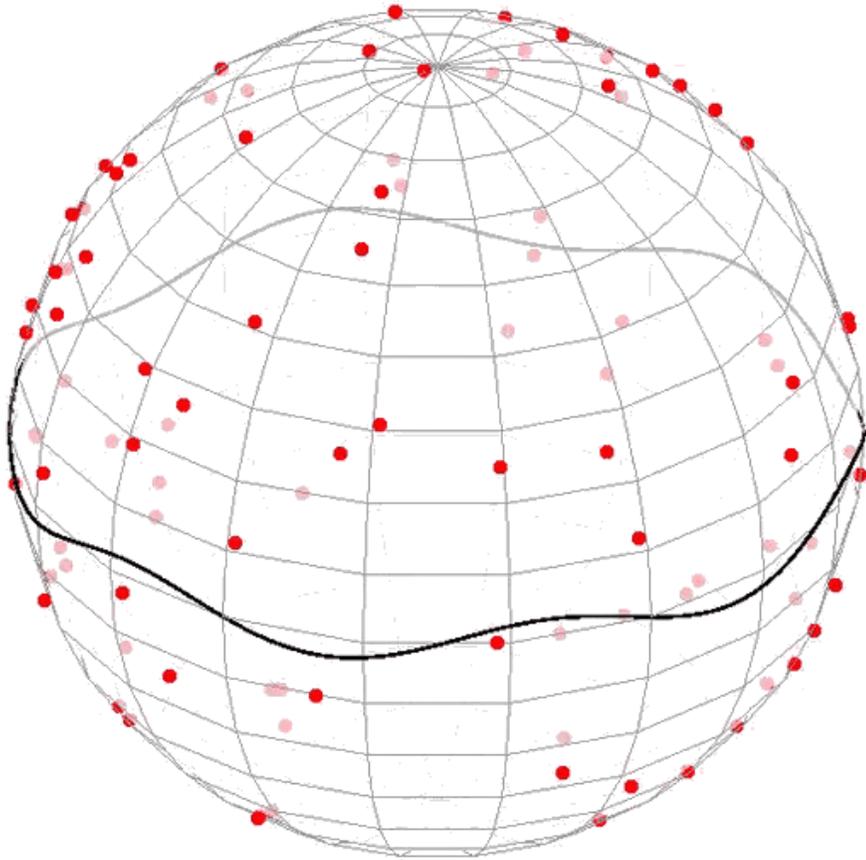
achieves almost global asymptotic tracking of the reference and local exponential convergence of the tracking error.

Sketch of Proof. We demonstrate that the intrinsic tracking error evolves governed by the dissipative mechanical system $\nabla_{\dot{e}} \dot{e} = - (dP(e) - k_D \dot{e})^\sharp$ and is thus almost globally asymptotically stable. Thus, $\dot{e}(t) \rightarrow 0_{TQ}$ and hence $\text{dist}(\dot{q}(t), \dot{q}_d(t)) \rightarrow 0$

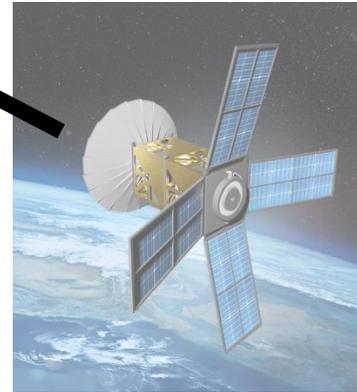
Almost Global Asymptotic Tracking on Spheres, e.g., \mathbb{S}^2

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TRACKING FOR
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$q \in \mathbb{S}^2$



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$p \in \mathbb{S}^2$



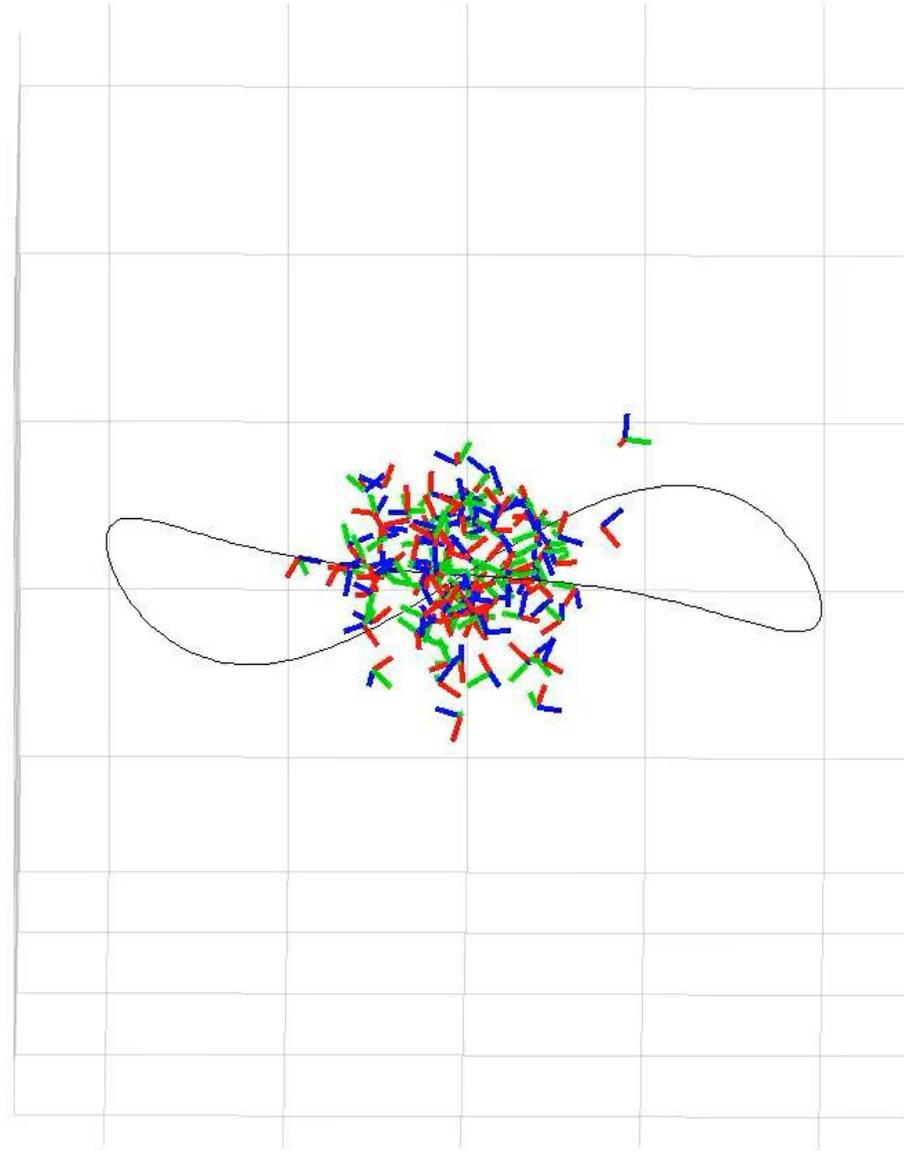
*Ascending
Technologies*

From a randomly selected initial state in $T\mathbb{S}^2$, the probability of failing to converge to the reference trajectory (position and velocity) is **exactly zero**.

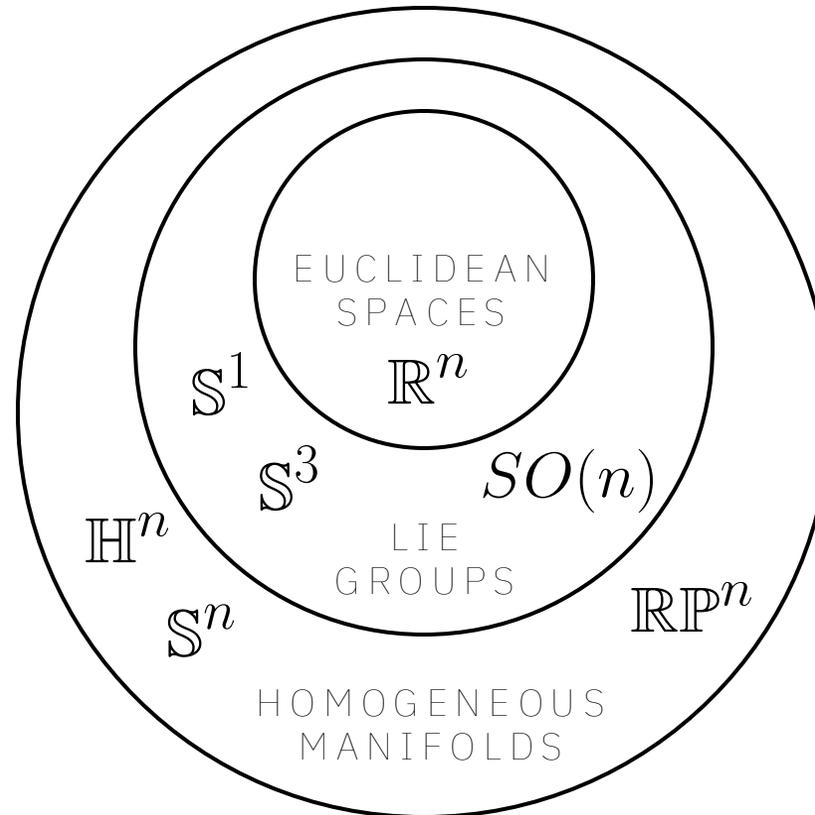
Tracking Control on Lie Groups, e.g., $\mathbb{R}^3 \times SO(3)$



*Omnicopter
on $SE(3)$*



We **systematically synthesize** smooth tracking controllers with **almost global convergence** on yet a **broader class of manifolds**.

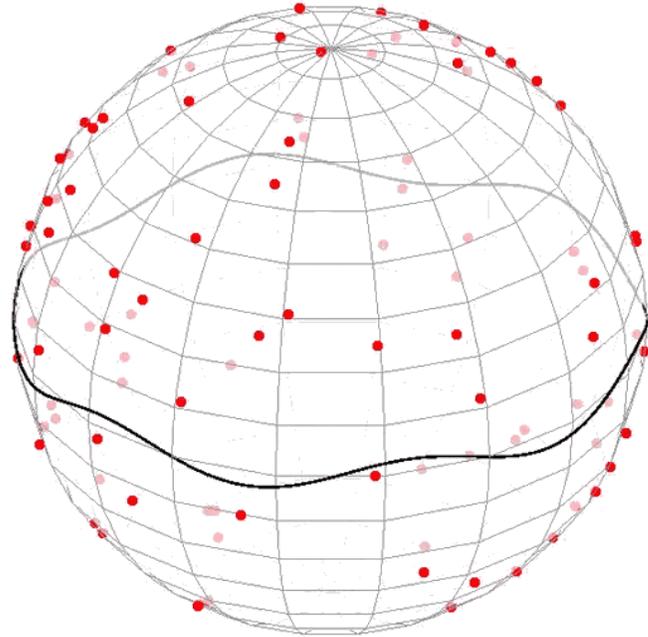


Because we guarantee **almost global asymptotic tracking**, the **probability** of a randomly selected initial state **failing to converge** to the reference trajectory is **exactly zero**.

“...the tracking problem on a Lie group is more closely related to tracking on \mathbb{R}^n than it is to the general Riemannian case, for which the group operation is lacking.” [2]

The case of **homogeneous Riemannian manifolds** is also similar to \mathbb{R}^n !

THANKS FOR LISTENING! QUESTIONS?



“P.S. - I’m on the academic job market!”



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