Reduction of Tracking Control Problems under Weak Symmetry

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***Abstract*—In high-dimensional state spaces, reinforcement learning (RL) algorithms often struggle with poor sample efficiency and inadequate policy exploration. To mitigate these issues in trajectory tracking problems, systems with continuous Lie group symmetries can be simplified without degrading performance by projecting the tracking dynamics onto a lower-dimensional quotient manifold, where states related by symmetry are replaced by relative errors. However, this classical notion of symmetry is in fact only a sufficient (but not necessary) condition for the projection of a tracking problem, indicating the potential for even greater reductions to be achieved. In this work, we investigate the projectability of tracking control problems using the recently-proposed notion of "weak symmetry". We prove that weak symmetry of a control system is a necessary *and* sufficient condition to exactly project the joint dynamics governing the actual and reference trajectories to the quotient space. Ultimately, this can significantly reduce the dimensionality of the state space, resolving bottlenecks in RL-based control and expanding the reduction of tracking control problems to systems that may not enjoy classical symmetries.**

I. Introduction

Modern robotic systems, ranging from aerial vehicles to articulated robotic arms, are governed by nonlinear dynamics evolving on smooth manifolds [1], [2]. Tracking control problems aim to drive such systems towards a given reference trajectory while maintaining stability [3] or optimality [4]. However, truly general solutions to analytical control design remain unavailable as they either require assumptions about the underlying dynamics (e.g., full actuation) [5] or focus on a single robot morphology [1]. Meanwhile, real-time optimal control methods like Model Predictive Control (MPC) struggle under the resource constraints of embedded hardware, necessitating the use of simplified schemes [6]. Alternatively, learning-based control addresses both of these bottlenecks; it exhibits remarkable adaptability to a wide range of problems while shifting the computational burden to offline training, thereby keeping real-time execution costs low [7].

However, as the system's dimensionality increases, learning algorithms often struggle with sample inefficiency, inadequate policy exploration, and poor generalization. Geometric symmetries inherent to the problem offer an opportunity to combat these issues, and symmetry-informed approaches such as Group Equivariant Convolutional Neural Networks [8] and Equivariant Reinforcement Learning (RL) [9] have demonstrated considerable improvements. Homomorphisms of Markov Decision Processes (MDPs) in discrete [10] and continuous [11] spaces can be used to establish lossless abstractions of learning problems by defining an equivalence relation over states that differ only by a symmetry transformation. When applied to RL-based tracking control—where the agent aims to learn a policy suitable for tracking arbitrary reference trajectories—the reduction of Lie group symmetries has drastically accelerated convergence to a high-performance policy [12].

A tracking problem can be formulated by evaluating the dynamics (governing the system's state) alongside an exact copy of the dynamics (governing the evolution of the reference under randomly sampled control inputs), which we thus call a "squared" system [12], [13]. When the underlying system exhibits classical symmetry, we may project this high-dimensional squared system to a lower-dimensional quotient manifold, replacing (at least some of) the actual and reference states with only the relative error between them [12] without degrading performance. Although necessary and sufficient conditions for the projectability of a given system (in this case, the squared system) are known [14], in tracking control, we desire such a condition on the original control system itself (and not on the squared system directly).

In this work, we show that the squared system describing the tracking problem can be exactly projected if and only if the underlying control system exhibits *weak symmetry* (a relaxed property recently proposed in [15]). Sec. II introduces the framework of symmetry definitions, and Sec. III shows that a wrench-controlled rigid body enjoys a weak $\mathrm{SE}_2(3)$ symmetry. Sec. IV presents our main contribution: that weak symmetry is necessary and sufficient to project squared systems, which we then apply to reduce the tracking control problem for the rigid body. Finally, Sec. V presents concluding remarks on the utility that weak symmetry offers as a principled mechanism for dimensionality reduction in RL-based tracking control.

II. Symmetry Preliminaries

Throughout this paper, let M be a smooth manifold, G a Lie group, $\mathfrak{g} = \mathrm{T}_e G$ the associated Lie algebra, and $\mathfrak{X}(M)$ denote the space of smooth vector fields on M. Let $\Phi : G \times M \to M$ be a free and proper group action, so that there exists a surjective submersion $\pi : M \to M/G$ taking each point $x \in M$ to its orbit in M/G. Given a map $\phi : X \times Y \to Z$, we will often use the partial application notation $\phi_x = \phi(x, \cdot) : Y \to Z$ for fixed $x \in X$, and likewise $\phi^y = \phi(\cdot, y) : X \to Z$ for fixed $y \in Y$. Now, we consider a control system described by a smooth map $f : \mathbb{U} \to \mathfrak{X}(M)$ such that the state $x(t)$ evolves under inputs $u(t) \in \mathbb{U}$ according to $\dot{x}(t) = f_{u(t)}\big(x(t)\big)$. For a more comprehensive introduction, see [2], [16]. We now introduce certain notions of symmetry of control systems.

Definition 1. The *residual* of $f : \mathbb{U} \to \mathfrak{X}(M)$ with respect to $\Phi : G \times M \to M$ is the map $\Delta^f : G \times \mathbb{U} \to \mathfrak{X}(M)$ given by

$$\Delta^f(g, u) = \mathrm{d}\Phi_{g^{-1}} \circ f_u \circ \Phi_g - f_u. \tag{1}$$

The residual is *autonomous* if there exists $\Delta^f : G \to \mathfrak{X}(M)$ such that $\Delta^f(g) = \Delta^f(g, u)$ for all $g \in G$ and $u \in \mathbb{U}$. We often use the convenient shorthand $\Delta^f_{(g,u)} := \Delta^f(g, u) \in \mathfrak{X}(M)$.

Traditionally, a symmetry of a control system is defined by the condition $f_u \circ \Phi_g(x) = \mathrm{d}\Phi_g \circ f_u(x)$ for all $g \in G$, $x \in M$, and $u \in \mathbb{U}$. Analyzing the residual's structure provides a natural framework for classifying different levels of symmetry, as it characterizes how much this "strong" symmetry is broken.

Definition 2. Consider a control system $f : \mathbb{U} \to \mathfrak{X}(M)$. A group action $\Phi : G \times M \to M$ is said to be:

- a *strong symmetry* [17] if the residual satisfies
$$\Delta^f_{(g,u)} = 0_{\mathfrak{X}(M)}, \tag{2}$$
- a *partial symmetry* [14] if there exists a *state-dependent generator* $\xi : G \times \mathbb{U} \times M \to \mathfrak{g}$ such that the residual satisfies
$$\Delta^f_{(g,u)}(x) = \mathrm{d}\Phi^x \circ \xi_{(g,u)}(x), \tag{3}$$
- or a *weak symmetry* [15] if there exists a *state-independent generator* $\xi : G \times \mathbb{U} \to \mathfrak{g}$ such that the residual satisfies
$$\Delta^f_{(g,u)}(x) = \mathrm{d}\Phi^x \circ \xi_{(g,u)}, \tag{4}$$

where each condition above must hold for all $g \in G$, $u \in \mathbb{U}$, and $x \in M$. In each case above, f is also said to be *strongly*, *partially*, or *weakly* *Φ-invariant*, respectively.

The preceding notions of strong and partial symmetry are well-established in the literature [14], [17], [18], whereas weak symmetry was recently introduced in [15] to capture a favorable middle ground between the other two notions. Note that the key distinction between symmetry types lies in the state-dependence of the generator. A partial symmetry permits the generator $\xi_{(g,u)}(x)$ to vary across the manifold, whereas a weak symmetry constrains $\xi_{(g,u)}$ to remain constant across all $x \in M$. Strong symmetry ($\xi_{(g,u)} \equiv 0$) is thus a special case of weak symmetry, which in turn is a special case of partial symmetry (as shown in Fig. 1).

In fact, partial symmetry is a necessary and sufficient condition to project a system onto a lower-dimensional quotient manifold [14]. Depending on the context, such a projection can be used to establish a semiconjugacy between control systems [19] or a continuous MDP homomorphism [11], illustrating the connection between symmetry and model reduction.

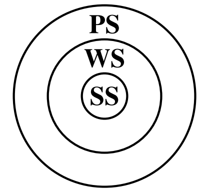

Fig. 1. Inclusion hierarchy of system symmetries. A system exhibiting a strong symmetry (SS) always exhibits a weak symmetry (WS), which in turn exhibits a partial symmetry (PS). The converse is generally false.

Proposition 1 (*cf.* [14, Prop. 2.7])**.** *Consider a control system $f : \mathbb{U} \to \mathfrak{X}(M)$ and a group action $\Phi : G \times M \to M$. Then, Φ is a partial symmetry of f if and only if there exists a unique projected system $\tilde{f} : \mathbb{U} \to \mathfrak{X}(M/G)$ such that $\mathrm{d}\pi \circ f_u = \tilde{f}_u \circ \pi$ for all $u \in \mathbb{U}$.*

III. Weak Symmetry of a Rigid Body

We consider an aerial vehicle consisting of a single rigid body subject to gravity and body-fixed control forces and torques. This system can be modeled as a left-invariant mechanical system on $\mathrm{SE}(3)$ subject to symmetry-breaking potential forces, and its dynamics evolving on $\mathrm{TSE}(3)$, can thus be derived using the Euler-Poincaré equations. Identifying $\mathrm{TSE}(3) \cong \mathrm{SE}(3) \times \mathfrak{se}(3)$ and $\mathfrak{se}(3) \cong \mathbb{R}^6$, the equations of motion take the form of the "manipulator equations",

$$\begin{bmatrix} \mathbb{J} & 0 \\ 0 & m\mathrm{I}_3 \end{bmatrix} \begin{bmatrix} \dot{\omega} \\ \dot{\nu} \end{bmatrix} + \begin{bmatrix} \omega_\times \mathbb{J} & 0 \\ 0 & m\omega_\times \end{bmatrix} \begin{bmatrix} \omega \\ \nu \end{bmatrix} + \begin{bmatrix} 0 \\ m\mathrm{g}R^\top \mathrm{e}_3 \end{bmatrix} = \begin{bmatrix} \mu \\ F \end{bmatrix},$$

where the state (R, p, ν, ω) consists of the pose $q = (R, p) \in \mathrm{SE}(3)$, the body-frame linear velocity $\nu \in \mathbb{R}^3$, and the body-frame angular velocity $\omega \in \mathbb{R}^3$. Here, $\omega_\times$ denotes the linear map satisfying $\omega_\times b = \omega \times b$ for all $b \in \mathbb{R}^3$. The control inputs $u = (\mu, F) \in \mathbb{R}^6$ represent the body-frame external moment and force, while $m > 0$ is the body's mass, $\mathbb{J}$ is the moment of inertia tensor, g is the magnitude of acceleration due to gravity, and e_3 is the vertical direction.

Expressing these dynamics in state-space form yields the control vector field $f : \mathbb{R}^6 \to \mathfrak{X}(\mathrm{SE}(3) \times \mathbb{R}^6)$:

$$f_u(R, p, \nu, \omega) = \begin{pmatrix} R\omega_\times \\ R\nu \\ \frac{1}{m}F - \mathrm{g}\, R^\top \mathrm{e}_3 - \omega \times \nu \\ \mathbb{J}^{-1}(\mu - \omega \times \mathbb{J}\omega) \end{pmatrix}, \tag{5}$$

where we identify our state manifold as $M = \mathrm{SE}(3) \times \mathbb{R}^6$. As mentioned above, this manifold is identified with $\mathrm{TSE}(3)$, which comes equipped with the "tangent Lie group" structure induced by the derivative of the group operation in $\mathrm{SE}(3)$. Since $\mathrm{TSE}(3)$ acts transitively on itself by left multiplication, the corresponding orbit map is a diffeomorphism. Hence, its differential is full rank and any residual can be expressed as the image of this differential. Therefore, by Def. 2, this action of $\mathrm{TSE}(3)$ is (trivially) a partial symmetry of f.

However, we will establish in Sec. IV that this partial symmetry is insufficient for projecting joint dynamics. Thus, we restrict our focus to the matrix Lie group $\mathrm{SE}_2(3)$, parameterized by a pose $(R, p) \in \mathrm{SE}(3)$ and a body-frame velocity $\nu \in \mathbb{R}^3$, namely,

$$\mathrm{SE}_2(3) = \left\{ \begin{bmatrix} R & R\nu & p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : R \in \mathrm{SO}(3),\ \nu, p \in \mathbb{R}^3 \right\}. \quad (6)$$

The group $\mathrm{SE}_2(3)$ was formalized in the context of inertial navigation by [20]. Since $\mathrm{SE}_2(3)$ is isomorphic to the subgroup of $\mathrm{TSE}(3)$ with zero angular velocity, it acts naturally on the state manifold M. In particular, we consider the action $\Phi : \mathrm{SE}_2(3) \times \mathrm{TSE}(3) \to \mathrm{TSE}(3)$ given by

$$\Phi_g(R, p, \nu, \omega) = \big(R_g R,\ p_g + R_g p,\ \nu + R^\top \nu_g,\ \omega\big) \quad (7)$$

for $g = (R_g, p_g, \nu_g) \in \mathrm{SE}_2(3)$. We claim that the rigid body system f exhibits a *weak symmetry* with respect to Φ.

To show this, we compute the residual $\Delta^f_{(g,u)} = \mathrm{d}\Phi_{g^{-1}} \circ f_u \circ \Phi_g - f_u$ for the $\mathrm{SE}_2(3)$ action. First, evaluating the vector field (5) at the transformed state $\Phi_g(x) = (R_g R,\ p_g + R_g p,\ \nu + R^\top \nu_g,\ \omega)$ gives

$$f_u\big(\Phi_g(x)\big) = \begin{pmatrix} R_g R \omega_\times \\ R_g(R\nu + \nu_g) \\ \frac{1}{m} F - \mathrm{g}\, R^\top R_g^\top \mathrm{e}_3 - \omega \times (\nu + R^\top \nu_g) \\ \mathbb{J}^{-1}(\mu - \omega \times \mathbb{J}\omega) \end{pmatrix}.$$

Next, we push this vector forward through the differential $\mathrm{d}\Phi_{g^{-1}}$. For some $\dot{x} = (\dot{R}, \dot{p}, \dot{\nu}, \dot{\omega}) \in \mathrm{T}M$, the differential acts as

$$\mathrm{d}\Phi_{g^{-1}}(\dot{x}) = \Big(R_g^\top \dot{R},\ R_g^\top \dot{p},\ \dot{\nu} - \dot{R}^\top R_g \nu_g,\ \dot{\omega}\Big).$$

The residual is therefore,

$$\Delta^f_{(g,u)}(x) = \big(0,\ \nu_g,\ \mathrm{g} R^\top (\mathrm{I}_3 - R_g^\top)\mathrm{e}_3,\ 0\big). \quad (8)$$

Now, consider the Lie algebra element

$$\xi(g) = \big(0,\ \nu_g,\ \mathrm{g}\mathrm{e}_3 - \mathrm{g} R_g^\top \mathrm{e}_3\big) \in \mathfrak{se}_2(3), \quad (9)$$

which represents a zero infinitesimal rotation, an infinitesimal position shift ν_g, and an infinitesimal velocity shift $\mathrm{g}\mathrm{e}_3 - \mathrm{g}R_g^\top \mathrm{e}_3$. Then, it is clear that for a state $x = (R, p, \nu, \omega)$, the infinitesimal generator map $\mathrm{d}\Phi^x : \mathfrak{se}_2(3) \to T_x M$ evaluated at $\xi(g)$ is exactly the residual in equation (8). Hence, we have recovered the autonomous residual as the infinitesimal generator $\Delta^f_{(g,u)}(x) = \mathrm{d}\Phi^x \circ \xi(g)$. Thus, by Def. 2, the rigid body system is weakly invariant under the $\mathrm{SE}_2(3)$-action Φ.

IV. Symmetries of a Squared System

As previewed in Sec. I, a tracking control problem is formulated by considering a dynamically feasible reference trajectory $\hat{x}(t) \in M$ with reference control input $\hat{u}(t) \in \mathbb{U}$, alongside the actual system state $x(t) \in M$ with true control input $u(t) \in \mathbb{U}$. Since both states evolve under the same dynamics $f : \mathbb{U} \to \mathfrak{X}(M)$, their joint evolution can be modeled as a single combined trajectory $(x, \hat{x}) \in M^2$ with a joint control input $(u, \hat{u}) \in \mathbb{U}^2$. This combined trajectory evolves under the *squared system* $f^2 : \mathbb{U}^2 \to \mathfrak{X}(M)^2 \subset \mathfrak{X}(M^2)$, defined as

$$f^2_{(u,\hat{u})}(x, \hat{x}) = \big(f_u(x), f_{\hat{u}}(\hat{x})\big). \quad (10)$$

The objective of tracking control is to make the actual state converge asymptotically to the reference trajectory. Thus, any symmetry of the overall tracking problem must transform the actual state and reference state by the exact same amount. As a result, we are interested in the *diagonal action*,

$$\bar{\Phi} : G \times M^2 \to M^2, \quad (g, x, \hat{x}) \mapsto (\Phi_g(x), \Phi_g(\hat{x})), \quad (11)$$

where it is clear that $x = \hat{x}$ if and only if $\Phi_g(x) = \Phi_g(\hat{x})$ (since Φ is free), preserving the tracking control objective.

We now show that the weak symmetry of the squared system (with respect to the diagonal action) is determined entirely by the weak symmetry of the underlying system.

Theorem 1. *The diagonal action $\bar{\Phi}$ is a weak symmetry of f^2 if and only if Φ is a weak symmetry of f with respect to an autonomous residual.*

Proof. The result follows from the fact that both the residual vector field and the infinitesimal generator map decouple over the product manifold. Specifically, for any $u, \hat{u} \in \mathbb{U}$, $(x, \hat{x}) \in M^2$, and $\xi \in \mathfrak{g}$, we have

$$\Delta^{f^2}_{(g,u,\hat{u})}(x, \hat{x}) = \big(\Delta^f_{(g,u)}(x),\ \Delta^f_{(g,\hat{u})}(\hat{x})\big), \quad (12)$$

$$\mathrm{d}\bar{\Phi}^{(x,\hat{x})}(\xi) = \big(\mathrm{d}\Phi^x(\xi),\ \mathrm{d}\Phi^{\hat{x}}(\xi)\big). \quad (13)$$

To show necessity, assume the diagonal action $\bar{\Phi}$ is a weak symmetry of f^2. By Def. 2, there exists a state-independent $\xi : G \times \mathbb{U}^2 \to \mathfrak{g}$ such that

$$\Delta^{f^2}_{(g,u,\hat{u})}(x, \hat{x}) = \mathrm{d}\bar{\Phi}^{(x,\hat{x})}(\xi_{(g,u,\hat{u})}). \quad (14)$$

Equating the components of (12)–(13) yields

$$\Delta^f_{(g,u)}(x) = \mathrm{d}\Phi^x(\xi_{(g,u,\hat{u})}), \quad (15)$$

$$\Delta^f_{(g,\hat{u})}(\hat{x}) = \mathrm{d}\Phi^{\hat{x}}(\xi_{(g,u,\hat{u})}). \quad (16)$$

Observe that the left-hand side of (15) is independent of $\hat{u}$. Since the action Φ is free, $\mathrm{d}\Phi^x$ is injective, forcing $\xi_{(g,u,\hat{u})}$ to be independent of $\hat{u}$. Symmetrically, (16) requires $\xi_{(g,u,\hat{u})}$ to be independent of u. Therefore, the generator must be autonomous, taking the form $\xi_{(g,u,\hat{u})} := \xi_g$, which proves that f possesses a weak symmetry with respect to an autonomous residual, $\Delta^f_g(x) = \mathrm{d}\Phi^x \circ \xi_g$ for all $x \in M$.

To show sufficiency, assume f possesses a weak symmetry with respect to an autonomous residual, that is $\Delta^f_g(x) = \mathrm{d}\Phi^x \circ \xi_g$ for a state-independent ξ_g. By substituting this into (12) and applying (13), we obtain

$$\Delta^{f^2}_{(g,u,\hat{u})}(x, \hat{x}) = \big(\mathrm{d}\Phi^x(\xi_g),\ \mathrm{d}\Phi^{\hat{x}}(\xi_g)\big), \quad (17)$$

$$= \mathrm{d}\bar{\Phi}^{(x,\hat{x})}(\xi_g). \quad (18)$$

Since the generator ξ_g is independent of the joint state $(x, \hat{x})$, then the diagonal action $\bar{\Phi}$ is a weak symmetry of f^2. ■

The preceding proof reveals a fundamental rigidity in the squared system: only autonomous, state-independent generators, which are exactly weak symmetries, can survive the lift to the product manifold, as formalized below.

Theorem 2. *The diagonal action $\bar{\Phi}$ is a partial symmetry of f^2 if and only if Φ is a weak symmetry of f with respect to an autonomous residual.*

Proof. If Φ is a weak symmetry of f with respect to an autonomous residual, Theorem 1 guarantees $\bar{\Phi}$ is a weak symmetry of f^2, which implies it is a partial symmetry.

Conversely, assume $\bar{\Phi}$ is a partial symmetry of f^2. By Def. 2, there exists a *state-dependent* generator $\xi : G \times \mathbb{U}^2 \times M^2 \to \mathfrak{g}$ such that the residual satisfies,

$$\Delta^{f^2}_{(g,u,\hat{u})}(x, \hat{x}) = \mathrm{d}\bar{\Phi}^{(x,\hat{x})}\big(\xi_{(g,u,\hat{u})}(x, \hat{x})\big). \tag{19}$$

As established in the proof of Theorem 1, the residual and generator maps decouple over M^2, yielding the component-wise equations:

$$\Delta^{f}_{(g,u)}(x) = \mathrm{d}\Phi^{x}\big(\xi_{(g,u,\hat{u})}(x, \hat{x})\big), \tag{20}$$

$$\Delta^{f}_{(g,\hat{u})}(\hat{x}) = \mathrm{d}\Phi^{\hat{x}}\big(\xi_{(g,u,\hat{u})}(x, \hat{x})\big). \tag{21}$$

Unlike in the previous proof, the generator here initially depends on both states x and $\hat{x}$. However, observe that the left-hand side of (20) is independent of $\hat{u}$ and $\hat{x}$. Since the action Φ is free, $\mathrm{d}\Phi^x$ is injective, forcing the argument $\xi_{(g,u,\hat{u})}(x, \hat{x})$ to also be independent of $\hat{u}$ and $\hat{x}$. Symmetrically, (21) dictates that $\xi_{(g,u,\hat{u})}(x, \hat{x})$ must be independent of u and x.

Consequently, the generator is independent of both the state and control inputs, taking the form $\xi_{(g,u,\hat{u})}(x, \hat{x}) := \xi_g$. The residual of f^2 is therefore generated by a state-independent Lie algebra element ξ_g, proving $\bar{\Phi}$ is a weak symmetry of f^2. By Theorem 1, this implies Φ is a weak symmetry of f with autonomous residual, completing the proof. ■

We stress that a partial symmetry of the base system f (such as the partial symmetry obtained from the transitive group law on $\mathrm{TSE}(3)$) does *not* guarantee a partial symmetry on the squared system. Instead, Theorem 2, combined with Prop. 1, proves that weak symmetry provides the exact structure required to project tracking control problems from the high-dimensional product space M^2 down to the relative error space M^2/G.

A. Revisiting the Rigid Body

Previously, we have shown that the rigid body possesses a weak symmetry under the $\mathrm{SE}_2(3)$-action Φ given by (7). We can express the projection map $\bar{\pi} : M^2 \to M^2/G$ as

$$\bar{\pi}(x, \hat{x}) = (\omega, \omega - R^\top \hat{R}\hat{\omega}, \hat{R}^\top R, \hat{R}^\top p - \hat{R}^\top \hat{p}, \hat{R}^\top R\nu - \hat{\nu}),$$

where $x = (R, p, \nu, \omega)$ and $\hat{x} = (\hat{R}, \hat{p}, \hat{\nu}, \hat{\omega})$. This projection map establishes the identification of the quotient space, $(\mathrm{SE}(3) \times \mathbb{R}^6)^2/\mathrm{SE}_2(3) \cong \mathbb{R}^3 \times \mathrm{TSE}(3)$. To simplify notation, we introduce the following error states in $\mathrm{TSE}(3)$:

$$\mathrm{E}_\omega := \omega - R^\top \hat{R}\hat{\omega}, \qquad \mathrm{E}_R := \hat{R}^\top R,$$
$$\mathrm{E}_\nu := \nu - R^\top \hat{R}\hat{\nu}, \qquad \mathrm{E}_p := \hat{R}^\top p - \hat{R}^\top \hat{p}.$$

Under these definitions, $\bar{\pi}$ maps the joint state to the quotient manifold using coordinates consisting of the true angular velocity ω and the relative error coordinates $(\mathrm{E}_\omega, \mathrm{E}_R, \mathrm{E}_\nu, \mathrm{E}_p)$.

By Theorem 2, this single-system weak symmetry with autonomous residual under the action of $G = \mathrm{SE}_2(3)$ guarantees the action $\bar{\Phi} : G \times M^2 \to M^2$ is a partial symmetry (and, in fact, a weak symmetry) of f^2. Hence, Prop. 1 implies the existence of a unique projected system

$$\widetilde{f^2} : \mathbb{U}^2 \to \mathfrak{X}\big(\mathbb{R}^3 \times \mathrm{TSE}(3)\big), \tag{22}$$

such that $\mathrm{d}\bar{\pi} \circ f^2_{(u,\hat{u})} = \widetilde{f^2}_{(u,\hat{u})} \circ \bar{\pi}$ for $u = (\mu, F)$ and $\hat{u} = (\hat{\mu}, \hat{F})$. In particular, we may verify that

$$\widetilde{f^2}_{(u,\hat{u})} \begin{pmatrix} \omega \\ \mathrm{E}_\omega \\ \mathrm{E}_R \\ \mathrm{E}_p \\ \mathrm{E}_R\mathrm{E}_\nu \end{pmatrix} = \begin{pmatrix} h_\mu(\omega) \\ h_\mu(\omega) + (\mathrm{E}_\omega)_\times \mathrm{E}_R^\top \hat{\omega} - \mathrm{E}_R^\top h_{\hat{\mu}}(\hat{\omega}) \\ \mathrm{E}_R(\mathrm{E}_\omega)_\times \\ \mathrm{E}_R\mathrm{E}_\nu - \hat{\omega}_\times \mathrm{E}_p \\ \frac{1}{m}\mathrm{E}_R F - \frac{1}{m}\hat{F} - \hat{\omega}_\times \mathrm{E}_R\mathrm{E}_\nu \end{pmatrix},$$

where $h_\mu(\omega) = \mathbb{J}^{-1}(\mu - \omega \times \mathbb{J}\omega)$ and we note that $\hat{\omega} = \mathrm{E}_R(\omega - \mathrm{E}_\omega)$ can easily be expressed in the quotient coordinates.

These dynamics are in agreement with [21, Eq. (5)] (although their equivalent results are formulated using world-frame velocities), illustrating that our principled approach using weak symmetry recovers the known error dynamics and reveals the underlying mechanism that generates them.

The projected system $\widetilde{f^2}$ captures the dynamics in the quotient space, which separate into the group invariant angular velocities $(\omega, \mathrm{E}_\omega)$ and the Lie group relative errors $(\mathrm{E}_R, \mathrm{E}_R\mathrm{E}_\nu, \mathrm{E}_p)$. While prior methods using strong symmetries could only factor out a four-dimensional subgroup $\mathrm{SO}(2) \ltimes \mathbb{R}^3$ [12], weak symmetry under $\mathrm{SE}_2(3)$ reduces the joint tracking control dynamics by nine dimensions, which we expect to significantly improve sample efficiency and tracking performance at convergence in RL-based control [12].

V. Conclusion

In this work, we considered a tracking control problem modeled using two independent trajectories of the same dynamics. We have proven that weak symmetry of the individual dynamics is the necessary and sufficient condition for the squared system governing the overall tracking problem to be projected under the diagonal action to a reduced system evolving on a lower-dimensional manifold. This establishes weak symmetry as a key condition for projection onto the quotient manifold and hence motivates further investigation of its role in control system reduction. Interesting directions for future research include investigating the reduction (under the diagonal action) of control systems composed of distinct subsystems (instead of a squared system, in which the subsystems are identical) and conditions for reduction in the context of observer design. In particular, we believe that MDP homomorphisms constructed using weak symmetries will enable yet greater sample efficiency and generalization when learning tracking controllers for robotic systems.

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